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MOTION OF A SPHERICAL HEAT SOURCE IN A MELTING MEDIUM

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The gravity-induced steady motion of a heat source boring through a solid medium is investigated. It is shown that the temperature of its surface is substantially nonuniform. Expressions are obtained for the maximum values of the radius and velocity of the source at which this temperature attains the assigned maximum allowable value.

One of the currently contemplated means for complete removal of radioactive wastes is "self-burial," when, due to the heat generated by their activity, the wastes bore through the rock in a high-melting container and become imbedded in it under gravity [1]. In this case a quasistationary temperature distribution is attained, and the heat source moves with a constant velocity [2–4]. The motion of a spherical source is considered in [3] on the assumption that the temperature of its surface is uniform. In the present work it is shown that the motion leads to a substantial nonuniformity of this temperature, whose maximum increases with increase in the radius.

We consider the heat source as a homogeneous sphere with the volume-averaged density and equivalent thermal conductivity:

$$\rho_i = \rho_2 + \left(1 - \frac{b}{R}\right)^3 (\rho_1 - \rho_2), \quad \frac{1}{k_i} = \frac{1}{k_1} + \frac{b}{R}\left(\frac{1}{k_2} - \frac{1}{k_1}\right). \tag{1}$$

We assume that the sphere is motionless and the solid medium has a constant velocity U. The temperature and thermal diffusivity of the solid phase and the melt surrounding the sphere are assumed to be identical.

The hydrodynamic equations for the thin layer of melt on the lower surface of the sphere $0 \le \theta \le \pi/2$ (see Fig. 1) in the approximation of lubrication theory have the form

$$\frac{\partial^2 v_\theta}{\partial r^2} = \frac{1}{\eta R}\frac{dp}{d\theta}, \quad \frac{\partial v_r}{\partial r}\sin\theta + \frac{1}{R}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) = 0. \tag{2}$$

Taking account of the boundary conditions

$$v_\theta(R, \theta) = 0, \quad v_r(R, \theta) = 0, \quad v_\theta(R + \delta, \theta) = U\sin\theta \tag{3}$$

and the continuity condition

$$2\int_R^{R+\delta} v_\theta dr = UR\sin\theta, \tag{4}$$

we find the distribution of velocities

$$v_\theta = U\frac{y}{\delta^*}\left[1 + \frac{3}{\delta^*}\left(1 - \frac{y}{\delta^*}\right)\right]\sin\theta, \quad v_r = -U\left(\frac{y}{\delta^*}\right)^2 \times$$

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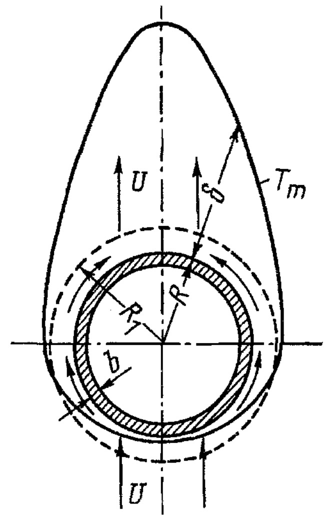

Fig. 1. Physical model and geometry of the problem.

$$\times\left\{\left(3-2\frac{y}{\delta^*}\right)\cos\theta-\frac{d\delta^*}{d\theta}\left[\frac{1}{2}+\frac{3}{\delta^*}\left(1-\frac{y}{\delta^*}\right)\right]\sin\theta\right\}, \tag{5}$$

and the stresses on the lower surface of the sphere

$$y=\frac{r}{R}-1,\quad \delta^*=\frac{\delta}{R}$$

$$\sigma_{rr}=-p,\quad \sigma_{r\theta}=\frac{3\eta U}{R\delta^{*2}}\sin\theta,\quad p=p_0+6\frac{\eta U}{R}\int_{\theta}^{\pi/2}\frac{\sin\theta}{\delta^{*3}}\,d\theta. \tag{6}$$

When $\delta^* \ll 1$, we may neglect the shear stress $\sigma_{r\theta}$ in comparison with the normal stress σ_{rr}. On the upper surface of the sphere we assume that

$$\sigma_{rr}=-p_0,\quad \sigma_{r\theta}=0. \tag{7}$$

The temperature distribution in the boundary layer on the lower surface is determined by the equation

$$v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{R}\frac{\partial T}{\partial\theta}=a\frac{\partial^2 T}{\partial r^2},\quad a=\frac{K}{\rho c_p}, \tag{8}$$

and the boundary conditions

$$T(R,\theta)=T_w(\theta),\quad T(R+\delta,\theta)=T_m,$$
$$-k\frac{\partial T}{\partial r}(R+\delta,\theta)=h\rho U\cos\theta,\quad h=h_m+c_p(T_m-T_\infty). \tag{9}$$

In [3] the temperature of the surface of the sphere T_w was assumed to be constant and problem (8), (9) was solved by the parametric method of boundary layer theory. We will solve this problem by the iterative method. As a first approximation we take a linear distribution in r:

$$T_1(r,\xi)=T_w(\xi)-[T_w(\xi)-T_m]\frac{y}{\delta^*},\quad \xi=\cos\theta.$$

Substituting it into the left-hand side of Eq. (8) and integrating taking account of the first two conditions of Eqs. (9), we find the following approximation:

$$T(r, \xi) = T_1(r, \xi) - \frac{3}{20}\frac{hS}{c_p}\left(1 - \frac{y}{\delta^*}\right)\beta y \times$$

$$\times\left\{\left[\xi - (1-\xi^2)\left(\frac{S'}{S} + \frac{5}{18}\delta^{*\prime}\right)\right]\left(1 + \frac{y}{\delta^*}\right) + \right.$$

$$+ \left[\xi + \frac{7}{3}(1-\xi^2)\left(\frac{S'}{S} - \frac{5}{42}\delta^{*\prime}\right)\right]\left(\frac{y}{\delta^*}\right)^2 -$$

$$\left. - \left[\frac{2}{3}\xi + (1-\xi^2)\frac{S'}{S}\right]\left(\frac{y}{\delta^*}\right)^3\right\}, \tag{10}$$

and we limit ourselves to it. The prime denotes derivatives with respect to ξ. The third condition in Eqs. (9) gives an equation for determining δ:

$$\frac{1}{8}(1-\xi^2)\delta^{*\prime} = \frac{1}{\beta\delta^*} = \varphi(\xi), \quad \varphi(\xi) = \frac{1}{S}\left[\xi\left(1 + \frac{7}{20}S\right) - \right.$$

$$\left. - \frac{1}{10}(1-\xi^2)S'\right], \quad S = \frac{c_p}{h}[T_w(\xi) - T_m], \quad \beta = \frac{UR}{a}. \tag{11}$$

Except for the small portion with $\xi < \varepsilon$, the thickness of the layer of melt varies fairly slowly. Therefore, we may neglect the first term in Eq. (11); then the thickness is determined by the formula

$$\delta^* = \frac{1}{\beta\varphi}. \tag{12}$$

Under the assumption that $S = \text{const} << 1$, it coincides with the result of [3]. On the portion with $\xi < e$ it is possible to neglect ξ^2 and φ in Eq. (11). In this case

$$\delta^* = \left[\delta^{*2}(0) - \frac{16}{\beta}\xi\right]^{1/2}. \tag{13}$$

Using the conditions of continuity and smoothness of the function $\delta(\xi)$ at $\xi = \varepsilon$ and assuming that $\varphi'(\varepsilon) = \varphi'(0)$, we find

$$\delta^*(0) = \frac{1}{\beta\varphi(\varepsilon)}\left[3 - 2\frac{\varphi(0)}{\varphi(\varepsilon)}\right]^{1/2},$$

$$\varepsilon = \frac{1}{\varphi'(0)}[\varphi(\varepsilon) - \varphi(0)], \quad \varphi(\varepsilon) = \frac{1}{2}\left[\frac{\varphi'(0)}{\beta}\right]^{1/3}. \tag{14}$$

According to Eqs. (10)-(12), we have the following expression for the heat flux density:

$$-k\frac{\partial T}{\partial r}(R, \xi) = \frac{kh\beta}{c_p R}\left\{\xi - \frac{1}{4}[(1-\xi^2)S]'\right\}. \tag{15}$$

We assume it to be valid for the entire lower surface $0 \le \xi \le 1$. The heat flux through the upper surface is ignored.

The temperature distribution in the interior of the source in the presence of axial symmetry satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T^i}{\partial r}\right)+\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial T^i}{\partial \xi}\right]=-\frac{q}{k_i}r^2. \quad (16)$$

Its solution, bounded at the points $r = 0$ and $\xi = \pm 1$, has the form

$$T^i(r, \xi) = T_m + \frac{4qR^2}{3k_i}\left[\frac{1}{8}(1-r^{*2}) + \sum_{n=0}^{\infty} C_n r^{*n} P_n(\xi)\right], \quad (17)$$

where $P_n(\xi)$ are Legendre polynomials; $r^* = r/R$. In this case the local Stefan number is equal to

$$S = \frac{4}{3}\nu \sum_{n=0}^{\infty} C_n P_n(\xi), \quad \nu = \frac{qR^2 c_p}{k_i h}. \quad (18)$$

From the boundary condition

$$k_i \frac{\partial T^i}{\partial r}(R, \xi) = \begin{cases} k\frac{\partial T}{\partial r}(R, \xi), & \xi > 0, \\ 0, & \xi < 0 \end{cases} \quad (19)$$

we obtain an infinite system of equations for determining the constants C_n:

$$S(0) = \frac{8}{3}\nu\gamma - 2, \quad C_n = \frac{n+1}{8\gamma}\sum_{n=0}^{\infty} C_m \int_0^1 P_m(\xi)\left[P_{n+1}(\xi) - P_{n-1}(\xi)\right] d\xi -$$

$$-\frac{2n+1}{8n}\left[\frac{3}{\nu\gamma}\int_0^1 \xi P_n(\xi)\, d\xi + \left(2 - \frac{3}{2\nu\gamma}\right) P_n(0)\right],$$

$$n = 1, 2, \ldots, \quad \gamma = \frac{k_i}{k\beta}. \quad (20)$$

Limiting ourselves to three termsin the infinite sum, we obtain

$$C_0 = \frac{2\gamma}{\Delta}\left(\gamma^2 + \frac{11}{64}\gamma + \frac{139}{10240}\right) - \frac{3}{2\nu},$$

$$C_1 = -\frac{\gamma}{2\Delta}\left(\gamma + \frac{3}{32}\right), \quad C_2 = -\frac{5\gamma}{32\Delta}\left(\gamma - \frac{49}{160}\right),$$

$$\Delta = \gamma^2 + \frac{27}{128}\gamma + \frac{33}{20480},$$

and the Stefan number takes the form

$$S(\xi) = S(1) + \frac{\nu\gamma}{48\Delta}(1-\xi)\left[47\gamma - \frac{51}{32} + 15\left(\gamma - \frac{49}{160}\right)\xi\right], \quad (21)$$

where

$$S(1) = \frac{8\nu\gamma}{3\Delta}\left(\gamma^2 - \frac{5}{32}\gamma + \frac{9}{640}\right) - 2. \quad (22)$$

With the aid of Eqs. (6), (7), and (12), we find the pressure force with which the melt acts on the sphere:

$$2\pi R^2 \int_{-1}^{1} p\xi d\xi = 6\pi a\eta\beta^4 J\,, \quad J = \int_{0}^{1} (1-\xi^2)\,\varphi^3 d\xi\,.$$

Equating it to the difference between the weight and Archimedes forces, we obtain an equation for determining γ:

$$J\beta^4 = \frac{2}{9}\frac{g}{a\eta}(\rho_i - \rho)\,R^3\,. \tag{23}$$

When $S(1) << 1$, the integrand Φ of the integral J has a sharp maximum at the point ξ_0 close to unity. Therefore the main contribution to the value of J is made by its neighborhood (ξ_1, ξ_2). In this neighborhood

$$\Phi = \Phi(\xi_0) + \frac{1}{2}\Phi''(\xi_0)(\xi - \xi_0)^2\,,$$

$$\xi_{1,2} = \xi_0 \pm \left[-2\frac{\Phi(\xi_0)}{\Phi''(\xi_0)}\right]^{1/2}, \quad \xi_0 = 1 + \frac{1}{2}\frac{S(1)}{S'(1)}\,,$$

$$\Phi(\xi_0) = -\frac{8}{27S'(1)\,S^2(1)}\,, \quad \Phi''(\xi_0) = \frac{64}{81}\frac{S'(1)}{S^4(1)}\,.$$

Thus

$$J = \frac{16\sqrt{3}}{81S'^2(1)\,S(1)}\,,$$

and, according to Eqs. (21) and (23),

$$S(1) = 0.923\frac{a\eta\beta^6}{g(\rho_i - \rho)\,R^7}\left[\frac{kh}{qc_p}f(\gamma)\right]^2, \tag{24}$$

$$f(\gamma) = \Delta/\left(\gamma - \frac{99}{992}\right).$$

Now, the value of γ can be found from Eq. (22), which at small values of $S(1)$ reduces to a cubic equation:

$$\gamma^3 - \frac{3}{4}\left(\frac{1}{\nu} + \frac{5}{24}\right)\gamma^2 - \frac{9}{128}\left(\frac{9}{4\nu} - \frac{1}{5}\right)\gamma - \frac{99}{81920\nu} = 0\,. \tag{25}$$

This equation has a single positive root. As ν increases, it decreases, and the velocity U increases.

The Stefan number assumes this largest value at the upper critical point

$$S(-1) = \frac{4}{3}\frac{\nu\gamma}{\Delta}\left(\gamma + \frac{3}{32}\right) \tag{26}$$

and increases with increase in ν. Equating $S(-1)$ to the value S_*, which is the maximum allowable value for a given container and which corresponds to the melting temperature of its surface T_*, we find, taking into account Eq. (25), the maximum value ν_*:

$$\nu_* = \frac{24S_*\Delta(\gamma_*)}{\gamma_*(32\gamma_* + 3)}\,, \tag{27}$$

$$\gamma_* = \frac{1}{2}\left[\frac{1}{S_*} + \frac{5}{32} + \left(\frac{1}{S_*^2} + \frac{11}{16S_*} - \frac{163}{5120}\right)^{1/2}\right].$$

Thus, the maximum velocity and radius are determined by the expressions

$$U_* = \frac{qR_*}{\rho h v_* \gamma_*}, \quad R_* = \left(\frac{k_i h}{q c_p} v_*\right)^{1/2} \tag{28}$$

and with allowance for Eq. (1) we have

$$R_* = \left[\frac{k_1 h v_*}{q c_p} + \frac{b^2}{4}\left(\frac{k_1}{k_2} - 1\right)^2\right]^{1/2} - \frac{b}{2}\left(\frac{k_1}{k_2} - 1\right). \tag{29}$$

To determine the boundary of the melting zone behind the sphere, we make the following assumptions. We consider a sphere of radius R_1, $R < R_1 < R + \delta(0)$ (shown by dashes in Fig. 1). In the region $R \le r \le R_1$, $\xi \le 0$ the temperature is assumed to be equal to $T_w(\xi)$, and with $r \ge R_1$, $\xi > 0$, except for a negligibly small neighborhood of $\xi = 0$, it may be assumed to be coincident with T_∞. Ignoring the disturbance of the velocity of the medium introduced by the sphere in the region $r \ge R_1$, $\xi < 0$, and the reverse release of the heat of the phase transition, we have an equation for the external temperature distribution:

$$-U\left(\xi\frac{\partial T^e}{\partial r} + \frac{1-\xi^2}{r}\frac{\partial T^e}{\partial \xi}\right) = \frac{a}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial T^e}{\partial r}\right) + \frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial T^e}{\partial \xi}\right]\right\}. \tag{30}$$

Its solution, decreasing with increase in r and bounded at the point $\xi = -1$, has the form

$$T^e(r, \xi) = T_\infty + \frac{1}{\sqrt{r}}\exp\left(-\frac{\beta r \xi}{2R_1}\right)\sum_{n=0}^{\infty} B_n K_{n+\frac{1}{2}}\left(-\frac{\beta r}{2R_1}\right)P_n(\xi), \tag{31}$$

where $K_{n+1/2}(x)$ are cylindrical functions of imaginary argument. From the boundary condition

$$T^e(R_1, \xi) = \tau(\xi) = \begin{cases} T_w(\xi), & \xi \le 0, \\ T_\infty, & \xi > 0 \end{cases} \tag{32}$$

we find the constants B_n:

$$B_n = \frac{(2n+1)\sqrt{R_1}}{2K_{n+\frac{1}{2}}\left(\frac{\beta}{2}\right)}\int_{-1}^{1}\exp\left(\frac{\beta}{2}\xi\right)[\tau(\xi) - T_\infty]P_n(\xi)\,d\xi.$$

By limiting ourselves to the leading term of the asymptotic expression

$$K_{n+\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{1/2}\exp(-x),$$

we simplify Eq. (31) to

$$T^e(r, \xi) = T_\infty + \frac{R_1}{r}[\tau(\xi) - T_\infty]\exp\left[-\frac{\beta}{2}(1+\xi)\left(\frac{r}{R_1} - 1\right)\right]. \tag{33}$$

Equating T_e to the melting temperature of the medium T_m and neglecting T_∞, we obtain an equation for the interface between the liquid and solid phases at $\xi \le 0$:

$$\frac{r(\xi)}{R_1} = [1 + \sigma(\xi)] \exp\left\{ -\frac{\beta}{2}(1+\xi)\left[\frac{r(\xi)}{R_1} - 1\right]\right\}, \quad \sigma(\xi) = hS(\xi)/c_pT_m. \tag{34}$$

The solution of this equation exists for a small exponent. Expanding it into a series and confining ourselves to the linear term, we find

$$\delta^*(\xi) = \frac{R_1}{R}\left\{1 + \frac{2\sigma(\xi)}{2 + \beta[1 + \sigma(\xi)](1+\xi)}\right\} - 1. \tag{35}$$

The value of R_1 can be found from the condition that this expression should coincide with Eq. (14) at $\xi = 0$.

As an example, we consider the case of imbedding of wastes in granite in a carbide-niobium container for the following data [3, 5]: $q = 130\,000$ W/m^3, $\rho_1 = 7800$ kg/m^3, $k_1 = 36$ W/(m·deg), $\rho_2 = 7800$ kg/m^3, $k_2 = 14$ W/(m·deg), $T_* = 3480$°C, $b = 0.01$ m; $\rho = 2700$ kg/m^3, $c_p = 1301$ J/(kg·deg), $k = 3$ W/(m·deg), $T_m = 1200$°C, $h_m = 585\,800$ J/kg, $\eta = 10$ kg/(m·sec).

According to Eqs. (14)-(35), we have $S_* = 1.382$, $\gamma_* = 0.937$, $\nu_* = 1.156$, $R_* = 0.719$ m, $U_* = 1.488 \cdot 10^{-5}$ m/sec = 469.34 m/year, $\beta = 12.527$, $S(1) = 1.5 \cdot 10^{-5}$, $S(0) = 0.889$, $\delta^*(1) = 1.2 \cdot 10^{-6}$, $\delta^*(0) = 0.490$, $\delta^*(-1) = 2.995$, $R_1/R = 1.378$, $\varepsilon = 0.109$.

The boundary of the melting zone is shown in Fig. 1 by a solid line. At the lower critical point the sphere is located virtually in direct contact with the solid medium and has a temperature equal to its melting temperature.

NOTATION

ρ, k, c_p, and a, density, thermal conductivity, specific heat, and thermal diffusivity of the medium; $\rho_{1,2}$ and $k_{1,2}$, densities and thermal conductivities of the wastes and the container material; η, dynamic viscosity of the melt; h_m, latent heat of melting; T_m and T_*, melting temperatures of the medium and he container material; q, R, and U, specific power, radius, and velocity of the heat source; T_w, surface temperature of the source; S, Stefan number; δ, thickness of the melt layer; p, pressure; v_r and v_θ, radial and azimuthal velocity components of the melt.

REFERENCES

1. J. J. Cohen, L. L. Schwartz and H. A. Tewes, Transactions of the American Nuclear Society, **18**, 194-195 (1974).
2. F. Jackson, Teploperedacha, No. 3, 10-13 (1965).
3. S. H. Emerman and D. L. Turcotte, Int. J. Heat Mass Transfer, **26**, 1625-1630 (1983).
4. M. K. Moallemi and R. Viskanta, J. Heat Transfer, Transactions of the ASME, **107**, No. 2, 186-195 (1985).
5. I. K. Kikoin (ed.), Tables of Physical Quantities: Handbook [in Russian], Moscow (1976).